TRINITY COLLEGE

Year 12 Methods Units 3 & 4
Test 4 2021

Section 1 Calculator Free

Logs & Continuous Random Variables

STUDENT'S NAME MARKING KEY [KRISZYK]

DATE: Friday 30th July **TIME:** 35 minutes **MARKS**: 34

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (5 marks)

Differentiate the following.

(a) $x^2 \ln x^2$ [2]

$$\frac{d}{dx} = 2x \ln x^2 + \frac{2x}{x^2} \times x^2$$

$$= 2x \ln x^2 + 2x$$

(b) $\ln\sqrt{\frac{4x+1}{x+3}}$ [3]

$$= \frac{1}{2} \ln \frac{4x+1}{x+3}$$

$$= \frac{1}{2}\left[\ln(4x+1) - \ln(x+3)\right]$$

$$\frac{d}{dx} = \frac{1}{2}\left(\frac{4}{4x+1}\right) - \frac{1}{2}\left(\frac{1}{x+3}\right)$$

2. (5 marks)

Determine the following.

(a) $\int \frac{2e^{4x}}{5-3e^{4x}} dx$ [2]

$$= \frac{-1}{6} \int \frac{-12e^{4x}}{5-3e^{4x}} \, dx$$

$$= \frac{-1}{6} \ln|5-3e^{4x}| + c$$

(b) $\int \tan\left(\frac{\pi\theta}{2}\right) d\theta$ [3]

$$= \int \frac{\sin\left(\frac{\pi}{2}\theta\right)}{\cos\left(\frac{\pi}{2}\theta\right)} \, d\theta$$

$$= \frac{-2}{\pi} \int \frac{-\sin\left(\frac{\pi}{2}\theta\right)\left(\pi/2\right)}{\cos\left(\frac{\pi}{2}\theta\right)} \, d\theta$$

$$= \frac{-2}{\pi} \ln\left|\cos\left(\frac{\pi}{2}\theta\right)\right| + c$$

3. (9 marks)

(a) Evaluate

(i) $\log 1000 - \log \frac{1}{100}$ [2]

$\log 10^3 - \log 10^{-2}$

$3\log 10 - (-2\log 10)$

$= 5$

(ii) $5^{2+\log_5 3}$ [2]

$\log_5 5^{2+\log_5 3} = \log_5 x$

$2 + \log_5 3 = \log_5 x$

$\log_5 25 + \log_5 3 = \log_5 x$

$\log_5 75 = \log_5 x$

$x = 75$

(b) Solve exactly $e^x = 7 + 8e^{-x}$ [3]

$e^x = 7 + \frac{8}{e^x}$

$(e^x)^2 = 7e^x + 8$

$(e^x)^2 - 7e^x - 8 = 0$

let $a = e^x$

$a^2 - 7a - 8 = 0$

$(a+1)(a-8) = 0$

$e^x = -1$

No Soln

$e^x = 8$

$\underline{\underline{x = \ln 8}}$

(c) Given $\log_a 3 = x$ and $\log_a 10 = y$, determine in terms of x and/or y the expression for $\log_a \frac{9a}{\sqrt{1000}}$ [2]

$\log_a 9 + \log_a a - \frac{1}{2}\log_a 10^3$

$2\log_a 3 + 1 - \frac{3}{2}\log_a 10$

$= 2x + 1 - \frac{3y}{2}$

4. (4 marks)

Consider the probability density function below:

$$p(x)=\begin{cases} ae^x & 0\le x\le 1 \\ ae & 1<x\le 2 \\ 0 & \text{otherwise} \end{cases}$$

Determine the value of a in terms of e.

$$\int_0^1 ae^x\,dx + \int_1^2 ae\,dx = 1$$

$$[ae^x]_0^1 + [xae]_1^2 = 1$$

$$ae^1 - ae^0 + 2ae^1 - ae = 1$$

$$2ae - a = 1$$

$$a(2e-1) = 1$$

$$a = \frac{1}{2e-1}$$

5. (6 marks)

The number of octaves (x) between two notes of frequencies f_1 (lower number) and f_2 (higher number) can be calculated from the formula:

$$x = \frac{1}{\log 2}\log\left(\frac{f_2}{f_1}\right)$$

(a) How many octaves are there between a note of frequency 110 Hz to one of 440 Hz? [2]

$$x = \frac{1}{\log 2} \times \log\left(\frac{440}{110}\right)$$

$$= \frac{\log 4}{\log 2}$$

$$= 2$$

(b) Ciara has a vocal range of 4 octaves. If her lower note of D has a frequency of 60 Hz, calculate her upper frequency. [4]

$$4 = \frac{1}{\log 2} \times \log\left(\frac{f_2}{60}\right)$$

$$4 = \frac{1}{\log 2}\left[\log f_2 - \log 60\right]$$

$$\log f_2 = 4 \log 2 + \log 60$$

$$f_2 = 16 \times 60$$

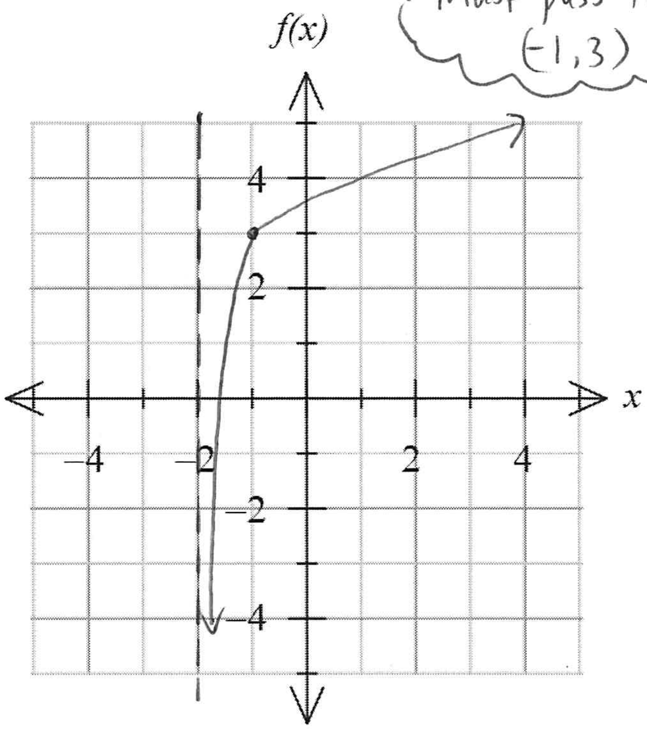
$$f_2 = 960 \text{ Hz}$$

6. (4 marks)

Sketch the following on the axes below, labelling at least 2 key features for each.

(a) $f(x) = \ln(x+2)+3$ [2]

Must pass through $(-1,3)$



Vertical asymptote $x = -2$ ✓

y int

$y = \ln(0+2)+3$

$y = \ln(2)+3$

$(0, \ln 2 + 3)$ ✓

or

x-int

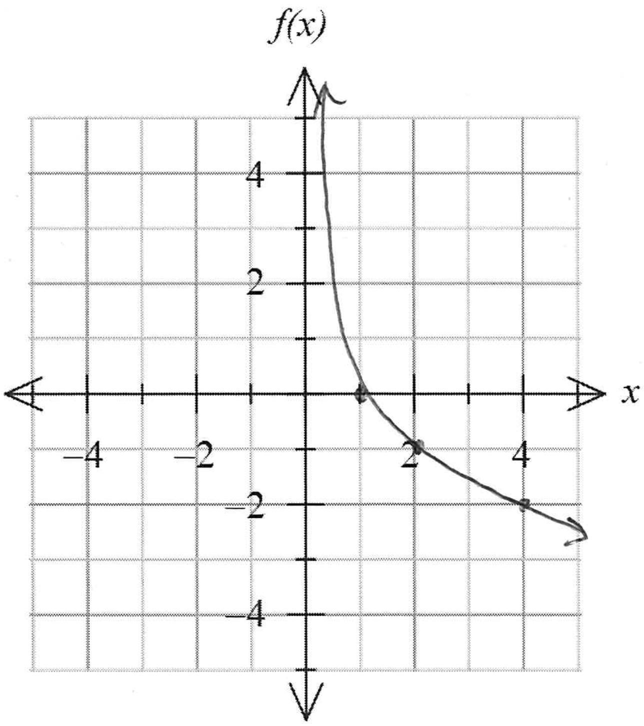
$\ln(x+2)+3 = 0$

$\ln(x+2) = -3$

$e^{-3} = x+2$

$x = \frac{1}{e^3} - 2$

(b) $f(x) = \log_2 \frac{1}{x}$ $= -\log_2 x$ [2]



x-int $(1,0)$ ✓

vertical asymptote $x=0$ ✓

must pass through

$(2,-1)$ and $(4,-2)$

TRINITY COLLEGE

Year 12 Methods Units 3 & 4
Test 4 2021

Section 2 Calculator Assumed
Logs & Continuous Random Variables

STUDENT'S NAME MARKING KEY [KRISZYK]

DATE: Friday 30th July **TIME:** 20 minutes **MARKS:** 25

INSTRUCTIONS:

Standard Items:	Pens, pencils, drawing templates, eraser
Special Items:	Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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5. (3 marks)

Determine k to 4 decimal places, if $f(x)$ is the probability distribution function for the random variable X, where

$$f(x)=\begin{cases} ke^{k^2x} & -1\le x\le 1 \\ 0 & \text{elsewhere}\end{cases}$$

Solve $\int_{-1}^{1} ke^{k^2x} = 1$

$\left[\frac{ke^{k^2x}}{k^2}\right]_{-1}^{1} = 1$

$k = 0.4950$

6. (12 marks)

The lifetime, X , in tens of hours, of a battery has a cumulative distribution function $F(x)$ given by:

$$F(x)=\begin{cases}0 & x<1\\ \frac{4}{9}(x^2+2x-3) & 1\leq x\leq 1.5\\ 1 & x>1.5\end{cases}$$

(a) Determine $P(X\geq 1.2)$ [2]

$$P(X\geq 1.2) = 1-P(X\leq 1.2) = 1-F(1.2) = \frac{47}{75} \quad 0.6267$$

(b) Determine, in full, the probability density function of the random variable X [3]

$$f(x) = F'(x)$$

$$f(x) = \begin{cases}\frac{4}{9}(2x+2) & 1\leq x\leq 1.5\\ 0 & \text{for all other } x\end{cases}$$

✓ ✓ ✓

(c) Determine $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ [2]

12.59 $\quad E(X) = \frac{340}{27}$ ✓

0.2075 $\quad Var(X) = \frac{1210}{5832}$ ✓

−1 questions (c), (d) or (e) if not ×10

(d) Determine the median life of a battery. [2]

Solve $F(x) = \frac{1}{2}$

$$\frac{4}{9}(x^2+2x-3) = \frac{1}{2}$$ ✓

$$x = 1.26$$

∴ median is 12.6 hours ✓

A camping lantern runs on 4 batteries, all of which must be working. Four new batteries are put into the lantern.

(e) Determine the probability that the lantern will still be working after 12 hours. [1]

$$\left(\frac{47}{75}\right)^4 = 0.1542 \checkmark$$

The company who manufactures the battery releases a heavy-duty version which has a 20% longer battery life.

(f) Determine the expected value and variance of the heavy-duty battery life. [2]

$$Y = 1.2X$$

$$E(Y) = 1.2 \times \frac{34}{27}$$

$$= \frac{68}{45} \times 10 \quad \checkmark$$

$$= \frac{136}{9} \quad [15.\dot{1}]$$

$$Var(Y) = (1.2)^2 \times \frac{121}{5832}$$

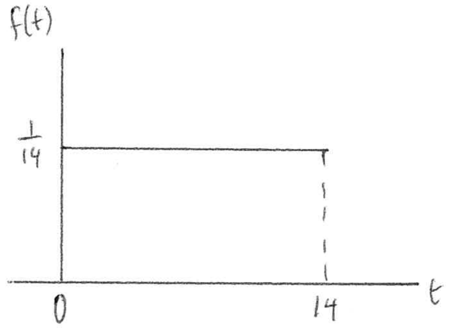
$$= \frac{121}{4050} \times 10 \quad \checkmark$$

$$= \frac{121}{405} \quad [0.2988]$$

7. (10 marks)

Mark catches the train every day (Monday to Friday) to school. His arrival time at the train station is uniformly distributed between 7.48 am and 8.02 am. Let the random variable T be the number of minutes Mark arrives at the train station after 7.48 am. The train he needs to catch always leaves at 8.00 am.

(a) State the probability density function for T and sketch it below. [3]



$$f(t) = \begin{cases} \frac{1}{14} & \text{for } 0 < t < 14 \\ 0 & \text{for all other } t \end{cases}$$

(b) Determine the probability that Mark arrives at the train station at 7:52 am [1]

0

(c) Determine the probability that Mark arrives at the train station before 7.58 am. [1]

$P(T<10) = \frac{10}{14}$ $[0.7143]$

(d) At 7.50 am Mark has yet to arrive at the station, what is the probability that he misses the train? [2]

$P(T>12 \mid T>2) = \frac{2}{12}$

$= \frac{1}{6}$ $[0.1667]$

(e) Determine the probability Mark misses the train more than once next week. [3]

$X \sim B(5, \frac{1}{7})$

$P(X>1) = 0.1518$